Some discussions of D. Fearnhead and D. Prangle's Read Paper "Constructing summary statistics for approximate Bayesian computation: semi-automatic approximate Bayesian computation"

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Abstract

This report is a collection of comments on the Read Paper of Fearnhead and Prangle (2011), to appear in the Journal of the Royal Statistical Society Series B, along with a reply from the authors.

1 A universal latent variable representation (C. Andrieu, A. Doucet and A. Lee)

Exact simulation to tackle intractability in model based statistical inference has been exploited in recent years for the purpose of exact inference Beaumont (2003); Beskos et al. (2006); Andrieu and Roberts (2009); Andrieu et al. (2010) (see Gouriéroux et al. (1993) for earlier work). ABC is a specialisation of this idea to the scenario where the likelihood associated to the problem is intractable, but involves an additional approximation. The authors are to be thanked for a useful contribution to the latter aspect. Our remarks to follow are presented in the ABC context but apply equally to exact inference. A simple fact which seems to have been overlooked is that sampling exactly $Y \sim f(y|\theta)$ on a computer most often means that $Y = \phi(\theta, U)$ where U is a random vector of probability distribution $D(\cdot)$ and $\phi(\cdot,\cdot)$ is a mapping either known analytically or available as a "black-box". The vector U may be of random dimension, i.e. $D(\cdot)$ may be defined on an arbitrary union of spaces (e.g. when the exact simulation involves rejections), and is most often known analytically - we suggest to take advantage of this latter fact. In the light of the above one can rewrite the ABC proxy-likelihood

$$\tilde{p}(y^*|\theta) = \int_{\mathsf{Y}} K(y, y^*) \times p(y|\theta) dy \quad ,$$

in terms of the quantities involved in the exact simulation of Y

$$\tilde{p}(y^*|\theta) = \int_{\mathsf{U}} K\left(\phi(\theta, u), y^*\right) \times D(u) du \quad .$$

In a Bayesian context the posterior distribution of interest is therefore

$$\tilde{p}(\theta|y^*) \propto \int_{\mathsf{U}} K\left(\phi(\theta,u),y^*\right) \times D(u)du \times p(\theta) \quad .$$

Provided that $D(\cdot)$ is tractable, we are in fact back to the usual, analytically tractable, "latent variable" scenario and any standard simulation method can be used to sample θ, U. Crucially one is in no way restricted to the usual approach where $U_i \overset{iid}{\sim} D(\cdot)$ to approximate the proxy-likelihood. In particular, for θ fixed, one can introduce useful dependence between $\phi(\theta,U_1), \phi(\theta,U_2), \ldots$ e.g. using an MCMC of invariant distribution $D(\cdot)$ started at stationarity Andrieu and Roberts (2009). The structure of $\tilde{p}(\theta,u|y^*)$ may however be highly complex and sophisticated methods may be required. One possible suggestion is the use of particle MCMC methods Andrieu et al. (2010) to improve sampling on the $U-$space, e.g. for a fixed value of θ estimate the proxy-likelihood $\int_{\mathsf{U}} K\left(y^*,\phi(\theta,u)\right) \times D(u)du$ unbiasedly using an SMC sampler Del Moral et al. (2006) targeting a sequence of intermediate distributions between $D(u)$ and $K\left(\phi(\theta,u),y^*\right) \times D(u)$ proportional to

$$K_j\left(\phi(\theta,u),y^*\right) \times D_j(u)$$

for $\{K_j(\cdot,\cdot), j = 1,\ldots,n-1\}$ and $\{D_j(\cdot), j = 1,\ldots,n-1\}$ and plug such an estimate in standard MCMC algorithms. Notice the flexibility offered by the choice of $\{K_j(\cdot,\cdot)\}$ and $\{D_j(\cdot)\}$ which can allow one to progressively incorporate both the dependence structure on U and the constraint imposed by $K(\cdot,\cdot)$. When $\phi(\cdot,\cdot)$ is known analytically, under sufficient smoothness conditions one can use an IPA Pflug (1996); Andrieu et al. (2005) approach to estimate e.g. gradients with respect to θ

$$\nabla_\theta \int_{\mathsf{U}} K\left(\phi(\theta,u),y^*\right) \times D(u)du \quad .$$

Again such ideas equally apply to genuine latent variable models and have the potential to lead to efficient exact inference methods in otherwise apparently "intractable" scenarios.

2 Summary-free ABC (S. Barthelmé, N. Chopin, A. Jasra and S.S. Singh)

We strongly believe that the main difficulty with ABC-type methods is the choice of summary statistics. Although introducing summary statistics may be sometimes beneficial Wood (2010), in most cases this induces a bias which is challenging to quantify. We thus welcome this important work on automatically choosing summary statistics. The fact remains that the optimality criterion proposed in the paper is a bit limiting; we want to approximate a full posterior distribution, not simply the posterior expectation. In addition, the proposed approach does not offer a way to monitor the bias induced by the optimal set of summary statistics, except by numerically comparing many alternative summary statistics, which is potentially tedious.

It is perhaps useful to note there now exist ABC methods that do not use summary statistics, at least for certain classes of models. The EP-ABC algorithm of Barthelmé and Chopin (2011) is a fast approximation scheme for ABC posteriors based on constraints on the form $\|y_i - y_i^\star\| < \epsilon$. It is typically orders of magnitude faster than Monte Carlo based ABC algorithm, whilst, in some scenarios, featuring an approximation error that is smaller, due to the absence of summary statistics. It is currently limited however to models such that the y_i may be simulated sequentially using some chain rule decomposition.

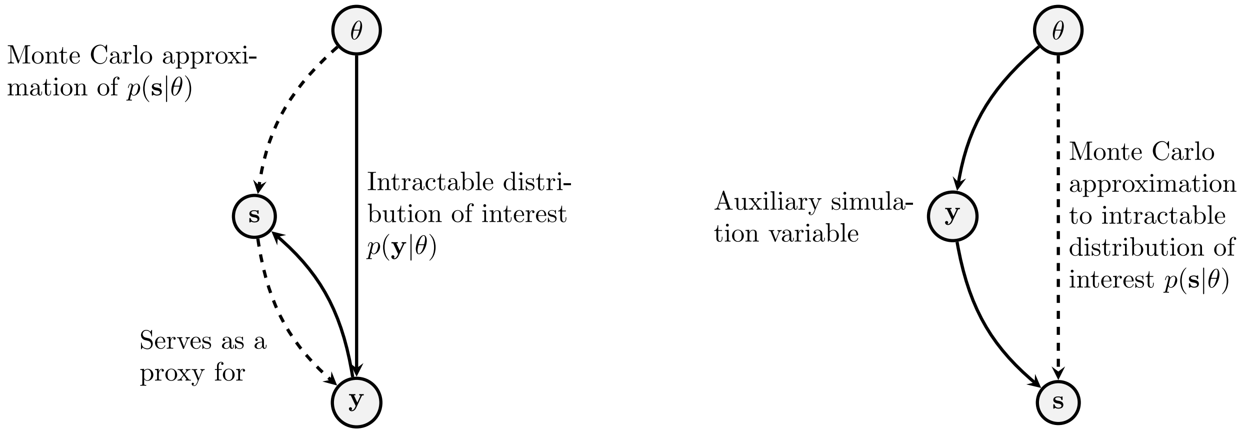
For hidden Markov models, "exact" ABC inference (i.e. not relying on either summary statistics or an approximation scheme) may be achieved as well, via the HMM-ABC approach of Dean et al. (2011); Dean and Singh (2011) (see also McKinley et al. (2009)), which show that an ABC posterior may be re-interpreted as the posterior of an artificial hidden Markov model, where the observations are corrupted with noise. This interpretation makes the remark of Wilkinson (2008) even more compelling: without summary statistics, an

ABC posterior may be interpreted as the correct posterior of a model where the *actual data* (as opposed to the summary statistics) are corrupted with noise. For instance, the Ricker model example, and with some adaptation for the Lokta-Volterra example of the read paper.

These two approaches already cover many ABC applications, and could be applied directly to three examples of the read paper: g-and-k distributions (EP-ABC), Lokta-Volterra processes (EP-ABC, HMM-ABC with a slight modification), and the Ricker model (HMM-ABC). We are currently working on extending this work in other dependence structures for the observations and we hope that others will also join us in this effort of removing summary statistics in ABC.

3 Inference on summary statistics: a mean or an end? (J. Cornebise and M. Girolami)

We congratulate the authors for their excellent article. We would like to suggest consideration of cases where it would make sense, from the perspective of statistical inference, to focus directly on $p(\theta|\mathbf{s})$, that is base inferences on the pre-processed, summarized data $\mathbf{s}$, rather than on the raw data $\mathbf{y}_{\text{obs}}$. Such a practice is standard in fields such as statistical discriminant analysis, pattern recognition, machine learning and computer vision, where pre-processing such as feature extraction (see e.g. Lowe, 2004), edge detection, and thresholding are routine, or in medical signal processing (e.g. MRI), where inference occurs on pre-processed output of the medical instrument. Wood (2010) focuses on qualitative descriptors of noisy chaotic dynamic systems presenting strong dependence on the initial conditions, with applications to ecological models: the primary interest for the user of these models are the characteristics of the trajectory (regularity, pseudo-period, maxima, extinction of the population, ...), not its actual path.


θ
Monte Carlo approximation of $p(\mathbf{s}|\theta)$
Intractable distribution of interest $p(\mathbf{y}|\theta)$
$\mathbf{s}$
Serves as a proxy for
$\mathbf{y}$
θ
Auxiliary simulation variable
$\mathbf{y}$
Monte Carlo approximation to intractable distribution of interest $p(\mathbf{s}|\theta)$
$\mathbf{s}$

(a) Classical use of ABC: inference based on the raw data $\mathbf{y}$, the summary statistics $\mathbf{s}$ serve to compute a Monte-Carlo estimate of $p(\mathbf{s}|\theta)$ as a proxy for the intractable likelihood $p(\mathbf{y}|\theta)$.

(b) Possible complementary use of ABC: inference based on the summarized data $\mathbf{s}$, the raw data $\mathbf{y}$ serves as an intermediate simulation step.

Figure 1: Graphical representation of the two possible uses of ABC: the roles of the data $\mathbf{y}$ and of the summary $\mathbf{s}$ are inversed. Plain lines represent distributions from which it is easy to sample; Annotated dashed lines represent logical relations.

Statistically speaking, as illustrated in the DAG of Figure 1, this is nothing but shifting the model one layer down the hierarchical model, permuting the role of $\mathbf{y}$ and $\mathbf{s}$ as auxiliary simulation variable and variable of interest, with the advantage of removing the proxy approximation: the summary statistics are not an approximation anymore, but the actual focus of interest. This is reminiscent of discriminative–generative modelling (see e.g. Xue and Titterington (2010) and Hopcroft et al. (2010)). The choice of those statistics then becomes either a modelling problem based on domain specific expertise or, drawing further the comparison with computer vision, a matter of sparse base construction as recently developed in compressed sensing (Candès and Wakin, 2008).

The only remaining layer of approximation is that of density estimation by the kernel K. Unfortunately, this kernel density estimation is only *asymptotically* unbiased, and is biased for *finite* sample size, the MH ratio in ABC-MCMC (Fearnhead and Prangle, 2011, Table 2) cannot be cast in the Expected Auxiliary Variable of Andrieu et al. (2007) extending Andrieu and Roberts (2009), not yet available but summarized in Andrieu et al. (2010), Section 5.1.

4 Parametric estimation of the summary statistics likelihood (J. Cornebise, M. Girolami and I. Kosmidis)

We would like to draw attention to the work of Wood (2010) which is of direct relevance to ABC despite it having been largely overlooked in the ABC literature. In Section 1.2 of the current paper the authors note that $K\left((S(\mathbf{y}) - \mathbf{s}_{\text{obs}})/h\right)$ is a Parzen-Rosenblatt density kernel. As has already been suggested in e.g. Del Moral et al. (2011) one can simulate R observations $\mathbf{y}_1, \ldots \mathbf{y}_R$ for a given value of θ and use the corresponding nonparametric kernel density estimate $\sum_r K\left((S(\mathbf{y}_r) - \mathbf{s}_{\text{obs}})/h\right)/(Rh^d)$ for $p(\mathbf{s}_{\text{obs}}|\theta)$. Wood (2010) suggests the *synthetic likelihood* by invoking the assumption of multivariate normality such that $\mathbf{s}_{\text{obs}} \sim \mathcal{N}(\boldsymbol{\mu}_\theta, \Sigma_\theta)$. Plug-in estimates of $\boldsymbol{\mu}_\theta$ and Σ_θ are obtained by the empirical mean $\hat{\boldsymbol{\mu}}_\theta^R$ and covariance $\hat{\Sigma}_\theta^R$ using the simulated statistics $S(\mathbf{y}_1), \ldots S(\mathbf{y}_R)$ yielding a parametric density estimate $\mathcal{N}(\mathbf{s}_{\text{obs}}; \hat{\boldsymbol{\mu}}_\theta^R, \hat{\Sigma}_\theta^R)$. This *synthetic likelihood* can then be used in an MCMC setting analagous to MCMC-ABC – and can similarly be used in IS-ABC and SMC-ABC settings. The convergence rate of the variance of the parametric density estimate is independent of the dimension of the summary statistics, which is in contrast to the nonparametric rate which suffers from the curse of dimensionality. This lower variance could improve mixing of the MCMC algorithm underpinning ABC, as already demonstrated in the pseudo-marginal approach to MCMC of Andrieu and Roberts (2009).

Of course Wood (2010) does not offer an automatic choice of the summary statistics: the user selects a (possibly large) set of summary statistics based on doman knowledge of the problem. This is similar to the way Section 3 offers to select the "transformations" $f(\mathbf{y})$, which are the first round of summary statistics. However, the relative weighting of each statistic is automatically inferred via the corresponding variance estimate. Could such a feature be of benefit in Semi-automatic ABC?

The assumption of multivariate normality on the distribution of the summary statistics plays a critical role in Wood's approach. He justifies it by: *i)* choosing polynomial regression coefficients as summary satistics and, most interestingly, *ii)* uses a pilot run to improve the normality of the statistics by quantile regression transformations – a preliminary step conceptually similar to the pilot ABC run of Section 3.

We conjecture that such transformations could allow for the use of parametric density estimation within Semi-automatic ABC, possibly benefitting from the increased convergence rate and making use of the variance of the sampled statistics. Additionally, we wonder if Theorem 4 could be modified to study the optimality of such transformed Gaussian statistics.

5 Automatic tuning of pseudo-marginal MCMC-ABC kernels (A. Lee, C. Andrieu and A. Doucet)

We congratulate the authors on a structured contribution to the practical use of ABC methods. We focus here on the conditional joint density

$$\bar{\pi}_{X,Y|\Theta}(x, y|\theta) = \pi_{Y|\Theta}(x|\theta)\bar{\pi}_{Y|X}(y|x),$$

which is central to all forms of ABC. Here x and y denote the simulated and observed data or summary statistics in ABC and $\bar{\pi}_{X|\Theta} = \pi_{Y|\Theta}$. In the article, $\bar{\pi}_{Y|X}(y|x) = K[(y - x)/h]$ and $\bar{\pi}_{Y|\Theta}(y|\theta) = \int \bar{\pi}_{X,Y|\Theta}(x, y|\theta)\mathrm{d}x \neq \pi_{Y|\Theta}(y|\theta)$ leads to the approximation. While neither $\bar{\pi}_{Y|\Theta}(y|\theta)$ nor $\pi_{Y|\Theta}(x|\theta)$ can be evaluated, the ability to sample according to $\pi_{Y|\Theta}(\cdot|\theta)$ allows for rejection, importance and MCMC sampling according to $\bar{\pi}_{\Theta,X|Y}(\cdot|y)$.

Algorithm 1 Rejuvenating GIMH-ABC

```
At time $t$, with $\theta_t = \theta$:
  1. Sample $\theta' \sim g(\cdot|\theta)$.
  2. Sample $z_{1:N} \sim \pi_{Y|\Theta}^{\otimes N}(\cdot|\theta')$.
  3. Sample $x_{1:N-1} \sim \pi_{Y|\Theta}^{\otimes(N-1)}(\cdot|\theta)$.
  4. With probability
       $$\min\left\{1, \frac{\pi(\theta')g(\theta|\theta')}{\pi(\theta)g(\theta'|\theta)} \times \frac{\sum_{i=1}^{N} \mathbf{1}_{B_h(z_i)}(y)}{1+\sum_{i=1}^{N-1} \mathbf{1}_{B_h(x_i)}(y)}\right\}$$
     set $\theta_{t+1} = \theta'$. Otherwise set $\theta_{t+1} = \theta$.
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The calibration of noisy ABC is then immediate. If $\tilde{y} \sim \bar{\pi}_{Y|X}(\cdot|y)$, then marginally $\tilde{y} \sim \bar{\pi}_{Y|\Theta}(\cdot|\theta^*)$ since $y \sim \pi_{Y|\Theta}(\cdot|\theta^*)$ for some $\theta^* \in \Theta$. Inference using $\bar{\pi}$ with $Y = \tilde{y}$ is then consistent with the data generating process although $\bar{\pi}_{\Theta|Y}(\cdot|\tilde{y})$ may not be closer to $\pi_{\Theta|Y}(\cdot|y)$ than $\bar{\pi}_{\Theta|Y}(\cdot|y)$.

The tractability of $\bar{\pi}_{\Theta,X|Y}$, whose unavailable marginal $\bar{\pi}_{\Theta|Y}(\cdot|y)$ is of interest puts ABC within the domain of pseudo-marginal approaches (Beaumont, 2003; Andrieu and Roberts, 2009), and the grouped-independence Metropolis-Hastings (GIMH) algorithm has been used in Becquet and Przeworski (2007). We present two novel MCMC-ABC algorithms based on recent work (Andrieu et al., 2012), and for simplicity restrict ourselves to the case $\bar{\pi}_{Y|X}(y|x) \propto \mathbf{1}_{B_h(x)}(y)$, where $\mathbf{1}_{B_h(x)}$ is the indicator function of a metric ball of radius h around x. These algorithms define Markov chains solely on Θ.

In the GIMH algorithm with N auxiliary variables, the state of the chain is $(\theta, x_{1:N})$ where $x_{1:N} := (x_1, \ldots, x_N)$ and at each iteration we propose new values $(\theta', z_{1:N})$ via $\theta' \sim g(\cdot|\theta)$ and $z_{1:N} \sim \pi_{Y|\Theta}^{\otimes N}(\cdot|\theta')$. Algorithm 1 presents an alternative to GIMH with the crucial difference in step 3, where GIMH would use the previously simulated values of $x_{1:N}$ instead of sampling $N-1$ new ones. This algorithm can have superior performance to the GIMH algorithm in some cases where the latter gets 'stuck'. Algorithm 2 involves a random number of simulations instead of fixed N, adapting the computation in each iteration to the simulation problem at hand. Data is simulated using both θ and θ' until a 'hit' occurs. It can be verified that the invariant distribution of θ is $\bar{\pi}_{\Theta|Y}(\cdot|y)$ for both algorithms. The probability of accepting the move $\theta \to \theta'$ after step 1 in Algorithm 1, as $N \to \infty$, approaches

$$\min\left\{1, \frac{\bar{\pi}_{\Theta|Y}(\theta'|y)g(\theta|\theta')}{\bar{\pi}_{\Theta|Y}(\theta|y)g(\theta'|\theta)}\right\}.$$

For Algorithm 2, this probability is exactly

$$\min\left\{1, \frac{\pi(\theta')g(\theta|\theta')}{\pi(\theta)g(\theta'|\theta)}\right\} \times \frac{\bar{\pi}_{Y|\Theta}(y|\theta')}{\bar{\pi}_{Y|\Theta}(y|\theta) + \bar{\pi}_{Y|\Theta}(y|\theta') - \bar{\pi}_{Y|\Theta}(y|\theta)\bar{\pi}_{Y|\Theta}(y|\theta')}.$$

Regarding the "automatic" implementation of ABC, Algorithm 1 could automate the use of N processors on a parallel computer or Algorithm 2 could be used to automatically adapt computational effort to the target of interest.

6 A new perspective on ABC (J.-M. Marin and C.P. Robert)

In this discussion paper, Fearnhead and Prangle do not follow the usual perspective of looking at ABC as a converging (both in N and h) approximation to the true posterior density (Marin et al., 2011b). Instead, they consider a randomised (or noisy) version of the summary statistics

$$s_{\text{obs}} = S(y_{obs}) + hx\,, \quad x \sim K(x)$$

Algorithm 2 1-hit MCMC-ABC

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At time $t$, with $\theta_t = \theta$:
    1. Sample $\theta' \sim g(\cdot|\theta)$.
    2. With probability $1 - \min\left\{1, \frac{\pi(\theta')g(\theta|\theta')}{\pi(\theta)g(\theta'|\theta)}\right\}$, set $\theta_{t+1} = \theta$ and go to time $t+1$.
    3. Sample $z_i \sim \pi_{Y|\Theta}(\cdot|\theta')$ and $x_i \sim \pi_{Y|\Theta}(\cdot|\theta)$ for $i = 1, \ldots$ until $y \in B_h(z_i)$ and/or $y \in B_h(x_i)$.
    4. If $y \in B_h(z_i)$ set $\theta_{t+1} = \theta'$ and go to time $t+1$.
    5. If $y \in B_h(x_i)$ set $\theta_{t+1} = \theta$ and go to time $t+1$.
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and they derive a calibrated version of ABC, i.e. an algorithm that gives “proper" predictions, but only for the (pseudo-)posterior based upon this randomised version of the summary statistics. This randomisation however conflicts with the Bayesian paradigm in that it seems to require adding pure noise to (and removing information from) the observation to conduct inference. Furthermore, Theorem 2 is valid for any value of h. We thus wonder at the overall statistical meaning of *calibration*, since even the prior distribution (corresponding to $h = +\infty$) is calibrated, while the most informative (or least randomised) case (ABC) is not necessarily calibrated. Nonetheless, the interesting aspect of this switch in perspective is that the kernel K used in the acceptance probability, with bandwidth h,

$$K((s - s_{\mathrm{obs}})/h)\,,$$

need not behave like an estimate of the true sampling density since it appears in the (randomised) pseudo-model.

As clearly stated in the paper, the ABC approximation is a kernel convolution approximation. This type of approximation has been studied in the approximation theory litterature. Typically, Light (1993) introduces a technique for generating an approximation to a given continuous function using convolution kernels. Also, in Levesley et al. (1996), it is constructed a class of continuous integrable functions to serve as kernels associated with convolution operators that produce approximations to arbitrary continuous functions. It could be eventually promising to adapt some of the techniques introduce in these papers.

Overall, we remain somehow skeptical about the “optimality" resulting from this choice of summary statistics as (a) practice—at least in population genetics (Cornuet et al., 2008)—shows that proper approximation to genuine posterior distributions stems from using a number of summary statistics that is (much) larger than the dimension of the parameter; (b) the validity of the approximation to the optimal summary statistics used as the actual summary statistics ultimately depends on the quality of the pilot run and hence on the choice of the summary statistics therein; this approximation is furthermore susceptible to deteriorate as the size of the pilot summary statistics grows; (c) important inferencial issues like model choice are not covered by this approach and recents results of ours (Marin et al., 2011a) show that estimating statistics are likely to bring inconsistent solutions in this context; those results imply furthermore than a naïve duplication of Theorem 3, namely based on the Bayes factor as a candidate summary statistic, would be most likely to fail.

In conclusion, we congratulate the authors for their original approach to this major issue in ABC design and, more generaly, for bringing this novel and exciting inferential method to the attention of the readership.

7 On the consistency of noisy ABC (C.P. Robert)

A discussion paper on the fast-growing technique of ABC techniques is quite timely, especially when it addresses the important issue of summary statistics used by such methods. I thus congratulate the authors on their endeavour.

While ABC has been gradually been analysed from a (mainstream) statistical perspective, this is one of the very first papers performing a decision-theoretic analysis of the factors influencing the performances of the method (along with, e.g., Dean et al., 2011). Indeed, a very interesting input of the authors is that ABC is considered there from a purely inferential viewpoint and calibrated for estimation purposes. The most important result therein is in my opinion the consistency result in Theorem 2, which shows that noisy ABC is a coherent estimation method when the number of observations grows to infinity. I however dispute the generality of the result, as explained below.

In Fearnhead's and Prangle's setting, the Monte Carlo error that is inherent to ABC is taken into account through the average acceptance probability, which collapses to zero when h goes to zero, meaning that $h = 0$ is a suboptimal choice. This is a strong (and valid) point of the paper because this means that the "optimal" value of h is not zero, a point repeated later in this discussion. The later decomposition of the error into

$$\text{trace}(A\Sigma) + h^2 \int x^{\mathrm{T}} A x K(x) \mathrm{d}x + \frac{C_0}{N h^d}$$

is very similar to error decompositions found in (classical) non-parametric statistics. In this respect, I do fail to understand the argument of the authors that Lemma 1 implies that a summary statistics with larger dimension also has larger Monte Carlo error: Given that $\pi(s_{\text{obs}})$ also depends on h, the appearance of h^d in eqn. (6) is not enough of an argument. There actually is a larger issue I also have against several recent papers on the topic, where the bandwidth h or the tolerance ϵ is treated as a given or an absolute number while it should be calibrated in terms of a collection of statistical and computational factors, the number d of summary statistics being one of them.

When the authors consider the errors made in using ABC, balancing the Monte Carlo error due to simulation with the ABC error due to approximation (and non-zero tolerance), they fail to account for "the third man" in the picture, namely the error made in replacing the (exact) posterior inference based on $\mathbf{y}_{\text{obs}}$ with the (exact) posterior inference based on $\mathbf{s}_{\text{obs}}$, i.e. for the loss of information due to the use of the summary statistics at the centre of the Read Paper. (As shown in Robert et al., 2011, this loss may be quite extreme as to the resulting inference to become inconsistent.) While the remarkable (and novel) result in the proof of Theorem 3 that

$$\mathbb{E}\{\theta | \mathbb{E}[\theta | \mathbf{y}_{\text{obs}}]\} = \mathbb{E}[\theta | \mathbf{y}_{\text{obs}}]$$

shows that $\mathbf{s}_{\text{obs}} = \mathbb{E}[\theta | \mathbf{y}_{\text{obs}}]$ does not loose any (first-order) information when compared with $\mathbf{y}_{\text{obs}}$, hence is "almost" sufficient in that weak sense, Theorem 3 only considers a specific estimation aspect, rather than full Bayesian inference, and is furthermore parameterisation dependent. In addition, the second part of the theorem should be formulated in terms of the above identity, as ABC plays no role when $h = 0$.

If I concentrate more specifically on the mathematical aspects of the paper, a point of the utmost importance is that Theorem 2 can only hold at best when θ is identifiable for the distribution $\mathbf{s}_{\text{obs}}$. Otherwise, some other values of θ satisfy $p(\theta | \mathbf{s}_{\text{obs}}) = p(\theta_0 | \mathbf{s}_{\text{obs}})$. Considering the specific case of an ancilary statistic $\mathbf{s}_{\text{obs}}$ clearly shows the result cannot hold in full generality. Therefore, vital assumptions are clearly missing to achieve a rigorous formulation of this theorem. The call to Bernardo and Smith, 1994 is thus not really relevant in this setting as the convergence results therein require conditions on the likelihood that are not necessarily verified by the distribution of $\mathbf{s}_{\text{obs}}$. We are thus left with the open question of the asymptotic validation of the noisy ABC estimator—ABC being envisioned as an inference method *per se*—when the summary variables are not sufficient. Obtaining necessary and sufficient conditions on those statistics as done in Marin et al. (2011a) for model choice is therefore paramount, the current paper obviously containing essential features to achieve this goal.

In conclusion, I find the paper both exciting and bringing both new questions and new perspectives to the forefront of ABC research. I am thus unreservedly seconding the vote of thanks.

8 On selecting summary statistics by post-processing (M. Sedki and P. Pudlo)

We congratulate the authors for their interesting and stimulating paper on ABC. Our attention was drawn to the regression building the new statistics in Section 3. Fearnhead and Prangle point similarities with the post-processing proposed by Beaumont et al. (2002). But they defend their algorithm on its ability to select an efficient subset of summary statistics. The main idea here is certainly to bypass the curse of dimensionality. E.g., in population genetics, a large number of populations commonly induce more than a hundred summary statistics with the DIYABC software of Cornuet et al. (2008).

Apart from Blum (2010), the widely used post-processing of Beaumont et al. (2002) has been little studied theoretically, although it significantly improves the accuracy of the ABC approximation. Actually, Beaumont et al. (2002) replace the θ's kept in the rejection algorithm with the residuals of a regression learning θ on the summary statistics. In the model choice settings (see, e.g., Robert et al., 2011), this post-processing uses a logistic regression predicting the model index, see Beaumont (2008). In both cases, it attempts to correct the discrepancy between the observed dataset and the simulated ones accepted by the ABC algorithm. We were intrigued by what would happen when postponing the variable selection criterion proposed in this paper until this post-processing.

Although a more detailed study is needed, we implemented two experiments: **(a)** one with a parameter estimation in the Gaussian family and **(b)** one with a model choice in the first population genetics example of Robert et al. (2011). We ran the classical ABC algorithm and used a Bayesian information criterion (BIC) during the local linear regression to select the relevant statistics. Then, we scanned once again the whole reference table drawn from the prior to find the nearest particles to the observation, considering only the subset of statistics selected by BIC. We ended with a local linear regression on this new set of particles. Numerical results are given in Figure 2 and show that applying BIC during Beaumont et al. (2002)'s post-processing is a promising idea.

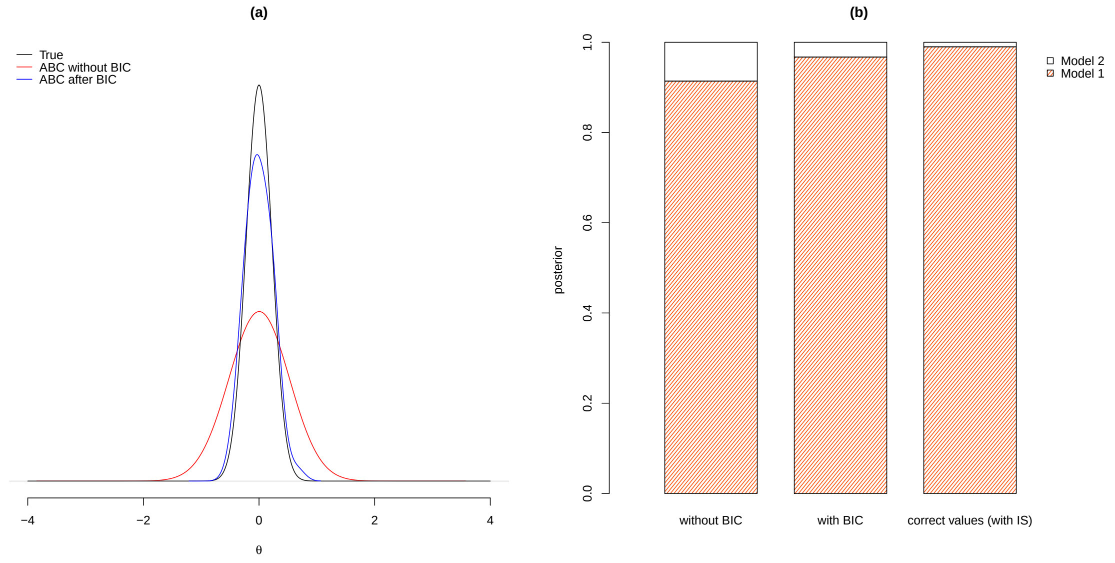

Figure 2: **(a)** Posterior density estimates in the first example. Prior over θ is Unif$(-5, 5)$, while X is a Gaussian vector of dimension 20, with independant components, $X_i|\theta \sim \mathcal{N}(\theta, 1)$. Summary statistics are $S_1 = \text{mean}(X_{1:20})$, $S_2 = \text{median}(X_{1:20})$, $S_3 \sim \text{Unif}(-5, 5)$ and $S_4 \sim \mathcal{N}(0, 1)$. Applying BIC here impoves the posterior density estimates by removing S_3 and S_4. **(b)** The model choice problem which is described in Robert et al. (2011) might be summed up the following way: considering three populations, we have to decide whether population 3 diverged from population 1 (Model 2) or 2 (Model 1). Among 24 summary statistics, BIC selects the two summary statistics LIK31 and LIK32 (see Tab. S1 of Robert et al. (2011)) which estimates genetic similarities between population 3 and the two other ones.

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